B-Splines

- Polynomial curves
- C^{k-1} continuity
- Cubic B-spline: C^2 continuity

Knots

- A sequence of scalar values $t_1, \ldots, t_{2k}$ with $t_i \neq t_j$ if $i \neq j$, and $t_i < t_j$ for $i<j$
- If t_i chosen at uniform interval (such as 1,2,3, …), than it is a uniform knot sequence

Control points, for k =3

- We can define a unique k degree polynomial F(t) with blossom f, such that $v_i = f(t_{i+1}, t_{i+2}, \ldots, t_{i+k})$
- The sequence of v_i for i [0,k] are the control points of a B-spline
- Evaluation of a point on a curve with f(t,t,t)
- Remark: no control points will lie on the curve!

Case k=2

- Knots: t_1,t_2,t_3,t_4
- Control points

$v_0 = f(t_1,t_2) = a_0 + a_1 (t_1+t_2)/2 + a_2 t_1 t_2$

$v_1 = f(t_2,t_3) = a_0 + a_1 (t_2+t_3)/2 + a_2 t_2 t_3$

$v_2 = f(t_3,t_4) = a_0 + a_1 (t_3+t_4)/2 + a_2 t_3 t_4$

Case k=3

- Knots: t_1,t_2,t_3,t_4 ,t_5,t_6
- Control points

$v_0 = f(t_1,t_2, t_3)$

$v_1 = f(t_2,t_3, t_4)$

$v_2 = f(t_3,t_4,t_5)$

$v_3 = f(t_2,t_5,t_6)$

Definition

- Given a sequence of knots, $t_1,\ldots t_{2k}$,
- For each interval $[t_i, t_{i+1}]$, there's a k^{th} degree parametric curve F(t) defined with corresponding B-spline control points $v_{i-k}, v_{i-k+1}, \ldots, v_i$
- If f() is the k-parameter blossom associated to the curve, then

Definition

- The control point are defined by $v_i = f(t_{j+1}, \ldots, t_{j+k})$, j=i-k, i-k, …, I
- The k-th degree Bézier curve corresponding to this curve has the control points:
  $p_j = f(\underbrace{t_i, t_i, \ldots, t_i}_{k-j}, \underbrace{t_{i+1}, t_{i+1}, \ldots, t_{i+1}}_{j})$, j = 0, 1, …, k
- The evaluation of the point on the curve at $t \in [t_i, t_{i+1}]$ is given by F(t) = f(t,t,…,t)

Relation between quadratic B-spline and Bézier curve

- K=2, limit on the ith interval, $t \in [t_i, t_{i+1}]$
- For the quadratic Bézier curve corresponding:
  $p0 = f(t_i, t_i)$ $p1 = f(t_i, t_{i+1})$ $p2 = f(t_{i+1}, t_{i+1})$
- For the B-Spline:
  $v_{i-2} = f(t_{i-1}, t_i)$ $v_{i-1} = f(t_i, t_{i+1})$ $v_i = f(t_{i+1}, t_{i+2})$
- And the interpolation:

$$f(t_i, t_i) = \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} v_{i-2} + \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} v_{i-1}$$

$$f(t_{i+1}, t_{i+1}) = \frac{t_{i+2} - t_{i+1}}{t_{i+2} - t_i} v_{i-1} + \frac{t_{i+1} - t_i}{t_{i+2} - t_i} v_i$$

B-splines or Bézier curves?

- Bézier curves are B-splines!
- But the control points are different
- You can find the Bézier control points from the B-spline control points
- In the case of a quadratic B-spline:
  p_0 is an interpolation between v_{i-2} and v_{i-1},
  $p_1 = v_{i-1}$
  p_2 is an interpolation between v_{i-1} and v_i

Advantage of B-splines over Bézier curves

- The convex hull based on m control points is smaller than for Bézier curve
- There is a better local control
- The control points give a better idea of the shape of the curve